Relativistic echo dynamics and the stability of a beam of Landau electrons

E Sadurní [1] and T H Seligman [1,2]

[1]Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México, Cuernavaca, Morelos, México.

[2]Centro Internacional de Ciencias, Cuernavaca, Morelos, México.

E-mail: `sadurni@fis.unam.mx, seligman@fis.unam.mx`

Abstract.

We extend the concepts of echo dynamics and fidelity decay to relativistic quantum mechanics, specifically in the context of Klein-Gordon and Dirac equations under external electromagnetic fields. In both cases we define similar expressions for the fidelity amplitude under perturbations of these fields, and a covariant version of the echo operator. Transformation properties under the Lorentz group are established. An alternate expression for fidelity is given in the Dirac case in terms of a 4-current. As an application we study a beam of Landau electrons perturbed by field inhomogeneities.

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Echo dynamics and fidelity decay have received considerable attention in recent years [1]. Their importance is underlined by the fact that fidelity is a standard benchmark in quantum information [2]. The relevance of these concepts is centered on non-relativistic quantum mechanics. It is rather surprising, that relativistic problems including quantum field theoretical ones have not been considered in this context.

In this letter we shall formulate echo dynamics and fidelity decay concentrating on the Dirac and Klein-Gordon equations, as the simplest cases. Field theoretical considerations or more complicated equations for particles with higher spins will be left for future work, though we hope that the concepts developed will readily generalize. On the other hand the inclusion of external fields is essential, as these will be the source of any reasonable perturbation. These fields will be assumed to result from a Lorentz covariant theory and couple minimally to the particle, although this last requirement can be relaxed in view of the intrest in systems such as the Dirac oscillator [3]. We shall limit our considerations both for simplicity, and because of the predominant practical importance to electromagnetic fields with their Abelian gauge structure. Let us stress that even if Dirac theory is thought to contain many particles already in first quantization, here we deal with the Dirac equation of one particle in the sense that just one set of space-time coordinates appears and we have to worry about evolution in one time only. In the light of past work [1], time scales of fidelity decay are of utmost importance when interpreting results. Here we would like to point out that relativity introduces a new time scale in terms of the speed of light, whose physical meaning is strongly related to the intensities of perturbations. This scale is given by $\hbar/mc^2$, m being the rest mass. Assuming that λ actually characterises the strength of the perturbation in energy units, we can compare to the standard timescale of fidelity decay $\hbar/\lambda$.

It is known [4] that relativistic effects appear when $\lambda > mc^2$, but also when the elapsed time t is long enough for the relativistic propagator to be significantly different from a delta function and therefore from its non relativistic counterpart. This means that for $t > \hbar/mc^2 > \hbar/\lambda$, quasi relativistic expansions are not useful, motivating thus a full relativistic treatment. Let us mention that our approach also includes the so called *echo kernel*, which generalizes the concept of the echo operator [1]. Its importance rests in the fact that it allows a perturbative treatment for arbitrarily long times, if the perturbation is weak enough, while the evolution operator does not.

We shall first discuss the problem for the Klein-Gordon equation and then proceed to the Dirac case. For the latter we can give an interesting alternate, though equivalent, formulation in terms of currents, which will be then used to discuss the perturbation of a Landau electron by field inhomogeneities. We close with some comments on possible generalizations. Let us consider two physical systems described by the Klein-Gordon equation with interaction. The systems will be considered to be minimally coupled to Lorentz covariant 4-vector potentials. One of the systems will be subject to a peturbation giving rise to a modified evolution of the wave function. We wish to give an expression for the corresponding fidelity. For simplicity we use units $\hbar = c = e = 1$ where e is the charge of the electron and c the speed of light. All space integrations will

be taken over R^3 unless indicated otherwise. The wave equation which describes the perturbed system contains the unperturbed one as a special case. It is given by

$$\hat{O}_{KG}(\epsilon)\phi(\epsilon) \equiv \left[D_\mu(\epsilon)D^\mu(\epsilon) + m^2\right]\phi(\epsilon) = 0 \tag{1}$$

where

$$D^\mu(\epsilon) = \partial^\mu + iB^\mu + i\epsilon A^\mu, \qquad D^\mu = D^\mu(0) \tag{2}$$

are the covariant derivatives with a "background" field B^μ and a perturbation A^μ modulated by ϵ. By means of the appropriate definition of the inner product in Klein-Gordon theory [5], namely

$$\langle\phi_1, t|\phi_2, t\rangle = i\int d^3x\left[\phi_1^*(x,t)\partial_0\phi_2(x,t) - (1\leftrightarrow 2)^*\right], \tag{3}$$

we can write the fidelity amplitude for a given initial condition $\phi(x)$. In fact, defining t as the 0 component of both (final) events x_μ and x'_μ, the fidelity amplitude f after a time t is given by

$$f(t) \equiv \langle\phi(\epsilon), t|\phi, t\rangle = \int d^3x' \int d^3x\phi^*(x)\phi(x')M(x_\mu, x'_\mu), \tag{4}$$

where an *echo kernel* M has been defined in terms of the Lorentz invariant propagator [6, 7] (or Feynman propagator) of equation (1). This propagator satisfies the equation

$$\hat{O}_{KG}(\epsilon)\Delta_\epsilon(x_\mu, x'_\mu) = i\delta^4(x_\mu - x'_\mu), \qquad \Delta = \Delta_{\epsilon=0} \tag{5}$$

and the echo kernel is found to be

$$M(x_\mu, x'_\mu) = \int d^3x''\left[\Delta^*_\epsilon(x''_\mu, x_\mu)\frac{\partial}{\partial t}\Delta(x''_\mu, x'_\mu) - \Delta(x''_\mu, x'_\mu)\frac{\partial}{\partial t}\Delta^*_\epsilon(x''_\mu, x_\mu)\right] \tag{6}$$

At this point a word must be said about the existence of an echo operator related to kernel (6). Despite the presence of M in (4), the lack of a standard hamiltonian formulation of the Klein-Gordon equation [8, 5] (*i.e.*, one with a hermitian Hamilton operator) forbids us to relate (6) in a direct way to a unitary evolution operator as in Schroedinger theory.

The Lorentz transformations for the Klein-Gordon fidelity amplitude and echo kernel are most simply derived by recalling the Lorentz invariance of the Feynman propagator Δ_ϵ and the transformation properties of the volume element. With this, through (4), it is easily shown that the effect of a boost in an arbitrary direction with associated Lorentz factor γ results in

$$f(t) = f(\gamma t'), \qquad \gamma = 1/\sqrt{1 - v^2} \tag{7}$$

where v is the boost velocity and t' is the observed time in the boosted frame. Thus the form of the amplitude is unchanged, while transformation acts exclusively on its argument as a time difference.

Next we discuss the fidelity amplitude and echo operator for the Dirac equation. Consider a system with background field and perturbation as above. It is now described by the equation

$$\hat{D}(\epsilon)\psi(\epsilon) \equiv \left[-i\gamma^\mu D_\mu(\epsilon) + m\right]\psi(\epsilon) = 0 \tag{8}$$

where $D_\mu(\epsilon)$ is given by (2) and γ^μ are Dirac matrices [9]. Since this theory allows a dynamical description in terms of a Hamiltonian, the concepts of fidelity and echo operator need little or no modification from the usual ones. The fidelity amplitude is thus given by

$$\langle\psi(\epsilon), t|\psi, t\rangle = \langle\psi(\epsilon)|U_t^{-1}(\epsilon)U_t|\psi\rangle = \langle\psi(\epsilon)|M_t|\psi\rangle \tag{9}$$

where $|\psi\rangle = |\psi, 0\rangle$ is our initial condition and $U(\epsilon)$ the unitary evolution operator. When H_D is set as the Dirac Hamiltonian associated to (8), U satisfies

$$H_D U(\epsilon) = -i\frac{\partial}{\partial t}U(\epsilon), \qquad U = U(\epsilon = 0). \tag{10}$$

Since a Schroedinger-like equation can be reached from (8) by factoring γ^0 from the Dirac operator $\hat{D}(\epsilon)$, we may prefer to handle evolution operators and propagators directly related to the Lorentz invariant equation (8) rather than (10). Thus, defining

$$U' = U\gamma^0, \qquad M_t' = (U_t'(\epsilon))^{-1}U_t'\gamma^0 = \gamma^0 M_t \tag{11}$$

we are led to the Lorentz invariant propagator

$$K_\epsilon'(x_\mu, x_\mu') = \langle x|U_t'(\epsilon)|x'\rangle \tag{12}$$

and the echo kernel

$$M'(x_\mu, x_\mu') = \langle x|M_t'|x'\rangle, \tag{13}$$

where t is the elapsed time (or time difference) between events x_μ and x_μ'. The fidelity amplitude is written in terms of the Dirac inner product as

$$f(t) = \langle\psi(\epsilon), t|\psi, t\rangle = \int d^3x \psi_\epsilon^\dagger(x, t)\psi(x, t). \tag{14}$$

It has the correct Lorenz transformation property (7) under application of a boost with the corresponding γ factor.

The Dirac equation allows another and more interesting approach to fidelity by means of a 4-vector which resembles a Dirac current. Consider a bilinear form in the wave functions given by

$$j_\mu(\epsilon) = \bar{\psi}(\epsilon)\gamma_\mu\psi, \qquad \bar{\psi} = \gamma^0\psi^\dagger. \tag{15}$$

The fidelity amplitude is obtained by integrating the 0 component of (15)

$$f(t) = \int d^3x j_0(\epsilon). \tag{16}$$

This displays clearly, that for $\epsilon = 0$, the fidelity amplitude becomes unity by conservation of probabilty. In fact, when the perturbation is present, the bilinear form (15) obeys a continuity-like equation

$$\partial^\mu j_\mu(\epsilon) = i\epsilon A^\mu j_\mu(\epsilon) \tag{17}$$

where the Abelian character of A has been used. Equation (17) indicates that the conservation law is now corrected by a source given exclusively in terms of the

perturbation applied to the system. Following this line of reasoning, (17) can be integrated over the space variables of some inertial observer to yield

$$\frac{df(t)}{dt} = \int_S \psi^\dagger(\epsilon) d\mathbf{s} \cdot \boldsymbol{\alpha}\psi - i\epsilon \int_V d^3x A^\mu j_\mu(\epsilon) \tag{18}$$

with V the volume of integration and S its boundary surface. The appearence of a boundary term can be related to fidelity loss when dealing with finite portions of space-time or with special boundary conditions for which wave functions do not vanish at infinity (e.g. unbound states). Its contribution is present independently of ϵ. Considering an infinite volume of integration V and a vanishing boundary term (bound states) we further simplify (18) using an expansion in ϵ to lowest order. In this approximation fidelity amplitude and fidelity respectively obey the differential equations

$$\frac{df(t)}{dt} = -i\epsilon \int d^3x A^\mu j_\mu = -i\epsilon \langle \gamma^0 \gamma^\mu A_\mu \rangle_I, f(0) = 1, \tag{19}$$

$$\frac{dF(t)}{dt} = -2\epsilon Re \left[i f(t) \langle \gamma^0 \gamma^\mu A_\mu \rangle_I \right], F(t) = |f(t)|^2 \tag{20}$$

where $\langle \cdot \rangle_I$ is the average with respect to ψ in the interaction picture. These two simple equations can be used to discuss some application.

We study Landau electrons perturbed by an inhomogeneous magnetic field. The system is described by the Dirac equation (8) with a homogeneous background field which, for some inertial observer, acquires the form $\mathbf{B} = \mathbf{H} \times \mathbf{r}$, $B_0 = 0$. *i.e.* a constant magnetic field of intensity H. Here $\mathbf{r}$ is the space part of x_μ for such an inertial observer. In this frame, when $t = x_0 = 0$ we switch on an additional field given by the perturbative static potential ϵA whose components are bounded such that the perturbation of strength ϵ is meaningful at all space-time points. If the system is stationary at $t < 0$, it will change to a non-stationary state at $t > 0$ and we would like to know how fidelity evolves.

The system at negative times is well known to be integrable. Its energy eigenstates are infinitely degenerate [10, 11]. A number of general results for fidelity amplitudes in the case of perturbed integrable systems have been established [1]. Some interesting features can be exploited here. The results for this setup will be valid not only for one electron propagating freely in the direction parallel to the field, but also for a beam of electrons under certain (physical) considerations. Although the treatment we have given does not contemplate many Dirac particles in the fashion of an extended configuration space [12, 13], we may consider the evolution to be described by a single time. The electromagnetic interactions between the constituents of the particle beam will be neglected, implying the absence of quantum field effects. This requires a wavelength in the propagating direction greater than Compton's as well as a low particle density in the beam.

With these physical restrictions and taking advantage of the infinite degeneracy of the unperturbed system, it is possible to accomodate an arbitray number of particles in a stationary state without violating the Pauli principle. We find that (20) (or equivalent expressions (12), (13) of [1]) yields a decay of fidelity dominated by a quadratic term

in t and with negligible oscillatory terms as we shall see in detail. When computing correlations in (12) and (13) from [1], the relevant term is given by

$$\int_0^t dt' \int_0^t dt'' \langle i|\tilde{V}(t')\tilde{V}(t'')|j\rangle$$
$$= 4\sum_j |\langle i|\gamma_0\gamma_\mu A^\mu|j\rangle|^2 \frac{\sin^2[(E_i - E_j)t/2]}{(E_j - E_i)^2} \tag{21}$$

where i and j indicate all quantum numbers including momentum, E_i, E_j being the corresponding energies. All degenerate levels will contribute to the dominant term of the sum in (21) even though $\langle i|\gamma_0\gamma_\mu A^\mu|i\rangle = 0$, which can be assumed without loss of generality [14]. The correlation can be approximated by

$$C(H,k)t^2 \equiv \left(\sum_{j:E_j=E_i} | < i|\gamma_0\gamma_\mu A^\mu|j > |^2\right) t^2 \tag{22}$$

where k is the $z-$momentum of the particle at $t < 0$. The H dependence enters through the unperturbed wave functions. We shall denote by $C(H)$ the coefficient in (22) when $k = 0$. When dealing with a beam of particles we may replace

$$C(H,k) = \sum_{n:E_n=E_i} w_n \left(\sum_{j:E_j=E_i} |\langle n|\gamma_0\gamma_\mu A^\mu|j\rangle|^2\right) \tag{23}$$

where w_n are statistical weights for states in the beam with energy E_i. Result (23) is not surprising from the point of view of integrability of the unperturbed system. Nevertheless it is remarkable in a theory of many fermions due to the infinite degeneracy of levels.

The result mentioned above can also be derived in the non-relativistic version of this problem. Lorentz transformations enter the game if we recognize that the momentum of the beam k is one of the control parameters (the others being H and ϵ) and that plane waves of arbitrary momentum can be obtained by applying boosts in the direction of propagation. From property (7), an increase of momentum k results in a Lorentz factor $1/\sqrt{1-v(k)^2}$, thus changing the rate of fidelity decay. These considerations lead finally to a decay of fidelity amplitude and fidelity given by

$$f(t) \sim 1 - \epsilon^2 C(H)\left(1 - (v(k))^2\right) t^2/2, \tag{24}$$

$$F(t) \sim 1 - \epsilon^2 C(H)\left(1 - (v(k))^2\right) t^2. \tag{25}$$

The variable t denotes the elapsed time as measured in the frame indicated above and the momentum dependent velocity is

$$v(k) = \frac{k}{\sqrt{k^2 + m^2}}. \tag{26}$$

Restoring our physical constants, the Compton wavelength restriction can be put as

$$k < 2mc, \quad v < 2c/\sqrt{5}, \quad \sqrt{1 - (v(k)/c)^2} > 1/\sqrt{5} \tag{27}$$

resulting in a lower bound for the magnitude of the decay rate. The upper bound for the momentum implies a maximum relativistic time delay. Allowing pair creation would slow down considerably the decay of fidelity, as long as $C(H)$ remains under control.

Summarizing, we have proposed a consistent formulation of echo dynamics and fidelity decay for the simplest relativistic wave equations. The perturbative regime, which has far reaching implications and applications in the non-relativistic case, yields an interesting behavior. We apply our formulation to a beam of Landau electrons propagating along a homogeneous magnetic field perturbed by bounded inhomogeneities.

Our treatment was limited to the discussion of a single particle with the corresponing single time line. Yet for systems of many particles, generalizations of the Dirac equation have been proposed; these involve in principle many *times* but a possibility for a single time evolution has been devised [12, 15]. Ultimately, composite relativistic systems can be treated along the lines we proposed here, since the Lorentz structure of evolution equations is not altered by extending configuration space (multiple space-time coordinates, one set for each particle) We also restricted our considerations to special relativity, but many of the results presented in this work can be generalized to curved spaces or "background" metrics by merely introducing covariant derivatives with the appropriate connections [16] and the corresponding transformation of spinors for the Dirac case [17]. Considering these facts our proposition for a relativistic formulation of fidelity and echo dynamics seems adequate or at least readily adaptable to the treatment of more general situations then the one we actually discuss. The issue of accelerated observers will be the subject of future work.

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